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# k-defects as compactons

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**Abstract**
We argue that topological compactons (solitons with compact support) may be quite common objects if *k*-fields, i.e., fields with nonstandard kinetic term, are considered, by showing that even for models with well-behaved potentials the unusual kinetic part may lead to a power-like approach to the vacuum, which is a typical signal for the existence of compactons. The related approximate scaling symmetry as well as the existence of self-similar solutions are also discussed. As an example, we discuss domain walls in a potential Skyrme model with an additional quartic term, which is just the standard quadratic term to the power two. We show that in the critical case, when the quadratic term is neglected, we get the so-called quartic $\phi^4$ model, and the corresponding topological defect becomes a compacton. Similarly, the quartic sine-Gordon compacton is also derived. Finally, we establish the existence of topological half-compactons and study their properties.

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## 1. Introduction

The present paper investigates the connection between *k*-fields, i.e., fields with dynamics governed by a nonstandard kinetic term and the appearance of a very special class of topological defects called *compactons*, that is, solitons which approach the vacuum value at a finite distance.

Classical field theories with a nonstandard kinetic term or, more generally, gradient term find more and more interesting applications in various branches of modern theoretical physics. Originally, such unusual gradients have been included to stabilize static solutions in some soliton models as, e.g., in the Skyrme [1] or Faddeev–Niemi models [2]. There the additional quartic term scales oppositely to the quadratic kinetic energy, such that the Derrick scaling argument against the existence of static finite-energy solutions is circumvented. A different strategy to circumvent Derrick's argument led to the investigation of Lagrangians where instead of the quadratic term one has a fractional power of the gradient. Moreover, due to the scaling symmetry, such models allow for the analytic calculation of exact chiral solitons [3, 4] or knot solitons [5–12].

Quite recently, models with generalized dynamics, i.e., with $k$-fields, have become particularly popular in cosmology. They have been proposed in the context of inflation leading to $k$-inflation, i.e., $k$-essence models [13–15], or as an alternative solution to the problem of dark matter [16, 17] (see, e.g., MOND, that is, modified Newtonian dynamics models [18]).

The behavior of topological defects (domain walls, vortices and monopoles) in models with a $k$-deformed kinetic part have also been studied [19, 20]. In general, it has been observed that the influence of the non-quadratic kinetic term leads to quantitative rather than qualitative differences. As we will show here, however, a deformation of the kinetic part can result in a more profound change of the properties of topological defects, namely in the existence of compactons.

As is suggested by its name, a compacton is a soliton with compact support. Compactons have originally been discovered as a special class of solitary waves in generalized versions of the KdV equation [21–25]. Further examples of compactons as, for instance, breather-like solutions are also known [26, 27].

Recently, compactons have been discussed in topologically nontrivial systems, as well. Such a topological compacton is an object which reaches the exact vacuum value at a finite distance. This class of topological defects naturally occurs in models with the standard quadratic kinetic scalar term provided that so-called $V$-shaped potentials (i.e., potentials which are not smooth at their minimum [28–33]) are considered. In particular, the left and right derivatives at the minimum do not vanish and the second derivative does not exist. In practice, this means that there is no mass scale in the system.

The most striking features characterizing compactons in standard models with $V$-shaped potentials are the parabolic approach to the vacuum value and the existence of an approximate scaling symmetry. Both effects are universal and do not depend on the particular form of the potential.

Interestingly, some $V$-shaped models can describe the pinning of topological defects to a boundary or an impurity.

Here, we show that compactons can emerge due to the nonstandard kinetic term even when the potential is assumed to be an analytical function of the scalar field $\xi$. In general, the appearance of compactons is a result of the mutual relation between the kinetic (more precisely spatial gradient term) and the potential part of the action.

The paper is organized as follows. In the following section, we give a general overview of the Bogomolny sector for $(1+1)$-dimensional field theories with the non-standard kinetic term. In section 3, the general relation between compactons and $k$-defects in $(1 + 1)$ dimensions is established. In particular, we find that the powerlike approach to the vacuum and the approximate scaling symmetry are present in our case, as well. Section 4 is devoted to the discussion of domain-wall-type compactons in the example of a scalar field theory embedded into a modified Skyrme model. Also the issue of stability is discussed in some detail. In section 5, we show that for slightly more general potentials half-compactons may exist. Half-compactons approach different vacuum values in a different manner, approaching some vacua at a finite distance, whereas they show the usual kink behavior (exponential approach) for other vacua. Section 6 contains our conclusions.

## 2. Bogomolny equation for non-quadratic Lagrangians in (1+1) dimensions

It is a well-known fact that for any scalar model in (1+1) dimensions with the usual quadratic kinetic term and arbitrary potential, there exists a Bogomolny sector defined by a certain first-order differential equation. Of course, in order to speak about the Bogomolny sector a nontrivial topology must exist. Therefore, we assume that there are at least two distinct ground states for the scalar field. The existence of such a first-order equation results in the following properties [34]:

(i) The second-order Euler–Lagrange equation of motion is identically satisfied by solutions of the Bogomolny equation.
(ii) The energy of the Bogomolny solution is completely determined by its topology.
(iii) The static energy-stress tensor vanishes identically.

In fact, these properties of the Bogomolny solutions are valid also in other, higher dimensional cases (two-dimensional vortices in the Abelian Higgs model, 't Hooft–Polyakov monopoles in the non-Abelian Higgs model, instantons of the four-dimensional Yang–Mills theory).

Of course, the models investigated here have a more complicated kinetic part. Therefore, it seems reasonable to look at this class of Lagrangians from a slightly more general point of view before analyzing domain walls in more specific models. First of all, it is important to understand the Bogomolny solution in these models.

Here, we show that one can easily find the Bogomolny equation for non-quadratic models. Additionally, we prove that the three properties mentioned above still hold.

### *2.1. Solution of the Bogomolny equation solves the equation of motion*

Let us consider the most general form of a Lorentz invariant Lagrangian depending on the scalar field and its first derivatives

$$L = L(v, \xi), \tag{1}$$

where

$$v \equiv \tfrac{1}{2}\partial_\mu \xi \partial^\mu \xi \tag{2}$$

(such models have been investigated, e.g., in [20, 35]). The equation of motion reads

$$\partial_\mu \left(L_v \partial^\mu \xi\right) - L_\xi = 0. \tag{3}$$

For static configurations it can be rewritten as

$$(L_w \xi_x)_x + L_\xi = 0, \tag{4}$$

where now

$$w \equiv -\tfrac{1}{2}\xi_x^2 \tag{5}$$

and $L$ is the static part of the Lagrangian. This equation can be integrated to

$$L + L_w \xi_x^2 = 0, \tag{6}$$

where the integration constant has been set to zero, which defines the Bogomolny sector in the one-dimensional case. By construction, all solutions of the first-order equation (6) satisfy the second-order field equation, as well. Moreover, due to the Lorentz invariance of the model we obtain a traveling (boosted) solution

$$\xi(x, t) = \tilde{\xi}(\gamma(x \pm \beta t)), \tag{7}$$

where $\tilde{\xi}$ is a solution of (6) and $|\beta| < 1$.

### 2.2. Space component of the static stress tensor vanishes identically

The corresponding energy-stress tensor has the form

$$T^{\mu\nu} = L_v \partial^\mu \xi \partial^\nu \xi - g^{\mu\nu} L. \tag{8}$$

In particular, one finds that

$$T^{01} = L_v \xi_t \xi_x, \tag{9}$$

$$T^{11} = L_v \xi_x^2 + L. \tag{10}$$

Of course, in the static case the component $T^{01} = 0$. Additionally, in the Bogomolny sector

$$T^{11} = L_w \xi_x^2 + L \equiv 0. \tag{11}$$

Therefore, for the solutions of the Bogomolny equation the space-like component of the static energy-stress tensor vanishes identically.

### 2.3. The energy of the solution depends only on its topology

The remaining non-vanishing component of the tensor gives the energy of the system. Here, we have

$$E = \int_{-\infty}^{\infty} T^{00}\,\mathrm{d}x = \int_{-\infty}^{\infty} \left(L_v \xi_t^2 - L\right) \mathrm{d}x. \tag{12}$$

In the Bogomolny sector,

$$E = \int_{-\infty}^{\infty} L_w \xi_x^2\,\mathrm{d}x. \tag{13}$$

On the other hand, equation (6) can be viewed as a complicated equation for $\xi_x$. A formal solution reads

$$\xi_x = \mathcal{F}(\xi). \tag{14}$$

Then,

$$E = \int_{-\infty}^{\infty} L_w|_{\xi_x=\mathcal{F}(\xi)} \mathcal{F}(\xi)\xi_x\,\mathrm{d}x = \int_{\xi(-\infty)}^{\xi(\infty)} L_w|_{\xi_x=\mathcal{F}(\xi)} \mathcal{F}(\xi)\,\mathrm{d}\xi. \tag{15}$$

In other words, for the solutions of the Bogomolny equation, the energy depends only on the boundary conditions for the scalar field, i.e., on the topology of the solution.

It is easy to note that this observation is true only if the scalar field reaches its vacuum value at infinity. Indeed, then the last equality holds and the energy is fixed by topology. In the opposite case, when the vacuum is approached at a finite distance, the situation changes. The global topology no longer defines the energy of the system. Instead, it is possible for a fixed total topological charge to construct solutions with various energies describing different collections of compactons and anti-compactons.

Therefore, compactons give a first example of the Bogomolny-type solutions for which there is no one-to-one correspondence between the global topology and energy.

## 3. *k*-fields and compactons

In the subsequent investigation we consider *k*-deformed Lorentz invariant Lagrangians. Nonetheless, as we analyze mainly the static configurations, our compactons may be solutions to other models, provided they have the same form in the static regime. Therefore they can also exist in some effective (non-Lorentz invariant) models, where the time evolution is governed by the usual second-time derivative [36].

*3.1. Quartic models*

We begin our analysis of the compactons in *k*-deformed Lagrangians with the *quartic* model

$$L = |\xi_\nu \xi^\nu| \xi_\mu \xi^\mu - U, \tag{16}$$

where instead of the standard quadratic kinetic term one deals with its quartic version. The potential $U$ is assumed to be a smooth function of the field $\xi$ with vanishing derivative at the local minimum.

The scalar field satisfies the following equation of motion:

$$4\partial_\mu[|\xi_\nu \xi^\nu| \xi^\mu] + U_\xi = 0. \tag{17}$$

*3.2. Parabolic approach*

For the static configurations, we easily derive the Bogomolny first-order equation

$$3\xi_x^4 = U. \tag{18}$$

In the vicinity of the local minimum located at $\xi_0$ the potential can be expanded into the Taylor series

$$U = U(\xi_0) + U'(\xi_0)(\xi - \xi_0) + \tfrac{1}{2}U''(\xi_0)(\xi - \xi_0)^2 + \cdots. \tag{19}$$

From the smoothness of the potential we get $U'(\xi_0) = 0$. Then, in the neighborhood of the minimum, a small perturbation $\delta\xi$ of the vacuum value of the scalar field

$$\xi = \xi_0 + \delta\xi, \tag{20}$$

obeys

$$(\delta\xi_x)^4 = \alpha(\delta\xi)^2, \tag{21}$$

where $\alpha^2 = U''(\xi_0)/6$. Here we consider the case with non-vanishing $U''(\xi_0)$. The obvious solution reads

$$\delta\xi = \left(\frac{\alpha}{4}\right)^2 x^2. \tag{22}$$

Therefore, in the quartic model the vacuum state is approached parabolically.

*3.3. Scaling symmetry*

Let us consider the full dynamical field equation near the local minimum

$$4\partial_\mu[|\xi_\nu \xi^\nu| \xi^\mu] + U''(\xi_0)(\xi - \xi_0) = 0. \tag{23}$$

One can check that if the field $\delta\xi$ obeys this equation then

$$\delta\xi_\lambda = \lambda^2 \delta\xi\left(\frac{x}{\lambda}, \frac{t}{\lambda}\right) \tag{24}$$

also is a solution of (23). This scaling symmetry is identical to the symmetry originally found in the $V$-shaped models. The symmetry is an approximate one as it exists only in an infinitesimal neighborhood of the local minimum.

In the case of the parabolic potential, such a scaling symmetry becomes exact. Then, in this *quartic harmonic oscillator* model one can construct self-similar solutions

$$\xi(x, t) = x^2 W\left(\frac{t}{x}\right). \tag{25}$$

This model can be viewed as the quartic counterpart of the signum-Gordon model discussed by Arodz.

As we see, the quartic model shares the universal features of the standard quadratic model with $V$-shaped potentials. However, while the standard compactons solve the field equations for all $x$ except the point where the vacuum value is reached (they are solutions in the weak sense), the compactons in the quartic model obey the field equations everywhere. This is true in spite the fact that the second derivative of the scalar field in general is not smooth at the boundary of the compacton. It is given by a step function. However, in the field equation of the quartic model this second derivative is multiplied by first derivatives, and the first derivative generically vanishes at the boundary of the compacton, rendering the field equation well defined even there.

### *3.4. Generalization*

A more general approach to the vacuum value, but still at finite distance, is realized in the following model:

$$L = |\xi_\nu \xi^\nu|^n \xi_\mu \xi^\mu - U, \tag{26}$$

where $n > -1/2$ and $n \neq 0$. This family of the Lagrangians belongs to the admissible theories considered by [35].

In fact, near the local minimum located at $\xi_0$ we get

$$\xi = \xi_0 + \frac{n}{1+n} \left( \frac{n^2 U''(\xi_0)}{2(n+1)^2(2n+1)} \right)^{\frac{1}{2n}} x^{\frac{n+1}{n}}. \tag{27}$$

Similarly, the corresponding approximate scaling symmetry is modified and takes the form

$$\delta\xi_\lambda = \lambda^{1+n} \delta\xi \left( \frac{x}{\lambda}, \frac{t}{\lambda} \right). \tag{28}$$

Again, this symmetry is exact if the potential takes the form

$$U = |\xi|^{n+1}. \tag{29}$$

The pertinent self-similar solutions can be found using the Ansatz

$$\xi(x,t) = x^{1+n} W\left(\frac{t}{x}\right). \tag{30}$$

Let us finally note that compactons should exist for all possible Lagrangians which asymptotically, for small values of $v = (1/2)\xi_\mu^2$, take the form given by expression (26). Then, the scalar field approaches the vacuum in a power-like manner and no exponential tail exists.

## 4. Example: Skyrme model with a new quartic term

### *4.1. Model*

As a particular example of a theory with the unusual kinetic term we use the Skyrme model modified by a new quartic part. Namely

$$\mathcal{L} = \frac{m^2}{2}\mathcal{L}_2 - M^2 \mathcal{L}_4 + \tilde{M}^2 \tilde{\mathcal{L}}_4 - \mathcal{L}_0, \tag{31}$$

where

$$\mathcal{L}_2 = \text{tr}(U^\dagger \partial_\mu U U^\dagger \partial^\mu U) \tag{32}$$

and

$$\mathcal{L}_4 = \mathrm{tr}[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2. \tag{33}$$

As in the standard Skyrme model, $U$ is a $SU(2)$ valued field living in (3+1)-dimensional spacetime. Moreover, $m$, $M$ and $\tilde{M}$ are parameters. The new quartic term is chosen in the following form:

$$\tilde{\mathcal{L}}_4 = |\mathcal{L}_2|\mathcal{L}_2. \tag{34}$$

In addition, we include a potential term $\mathcal{L}_0$ which will be specified below.

The matrix field can be parameterized by

$$U = \mathrm{e}^{\mathrm{i}\vec{\xi}\vec{\sigma}}, \tag{35}$$

where $\vec{\sigma}$ are the Pauli matrices and $\vec{\xi}$ is a three component real vector field. However, it is convenient to adopt a different parametrization

$$U = \mathrm{e}^{\mathrm{i}\xi\vec{n}\vec{\sigma}}, \tag{36}$$

where

$$\xi = |\vec{\xi}|, \qquad \vec{n} = \frac{\vec{\xi}}{|\vec{\xi}|} \tag{37}$$

and the unit vector field is expressed by a complex scalar field $u$ via stereographic projection

$$\vec{n} = \frac{1}{1+|u|^2}(u+u^*, -\mathrm{i}(u-u^*), |u|^2-1). \tag{38}$$

In terms of the new variables we find that

$$\mathcal{L}_2 = \xi_\mu \xi^\mu + 4\sin^2\xi \frac{u_\mu \bar{u}^\mu}{(1+|u|^2)^2} \tag{39}$$

$$\mathcal{L}_4 = 16\sin^2\xi \left( \xi_\mu \xi^\mu \frac{u_\mu \bar{u}^\mu}{(1+|u|^2)^2} - \frac{\xi^\mu u_\mu \xi_\nu \bar{u}^\nu}{(1+|u|^2)^2} \right) + 16\sin^4\xi \frac{(u_\mu \bar{u}^\mu)^2 - u_\mu^2 \bar{u}_\nu^2}{(1+|u|^2)^4}. \tag{40}$$

In order to consider domain walls, we assume that the complex field $u$ is trivial, i.e., it takes its vacuum value everywhere,

$$u = 0 \quad \Rightarrow \quad \vec{n} = (0, 0, 1). \tag{41}$$

As a consequence, we deal with the following Lagrangian

$$L = \frac{m^2}{2}\xi_\mu \xi^\mu + \tilde{M}^2 |\xi_\mu \xi^\mu| \xi_\mu \xi^\mu - L_0(\xi). \tag{42}$$

As we see, the appearance of the new quartic term qualitatively changes the Lagrangian describing the dynamics of domain walls. In contrast to the usual Skyrme model where the quartic Skyrme term identically vanishes if Ansatz (41) is assumed, the new quartic part does contribute to the Lagrangian.

In the subsequent analysis we consider two particular potentials, which in our parametrization are the well-known $\phi^4$ potential

$$L_0 = 3\lambda^2(\xi^2 - a^2)^2 \tag{43}$$

or the sine-Gordon potential

$$L_0 = \tfrac{3}{2}\lambda^2(1 - \cos\xi), \tag{44}$$

respectively. Here, $\lambda$ is a real constant. Moreover, we restrict the spacetime to (1+1) dimensions. From the point of view of the modified Skyrme model it means that physical quantities like energy density or energy are given per unit area.

Note that this model slightly differs from the class of Lagrangians which have been recently analyzed in [19, 20]. In fact, the model considered by these authors reads

$$L = \frac{m^2}{2}\xi_\mu\xi^\mu - \tilde{M}^2(\xi_\mu\xi^\mu)^2 - L_0(\xi) \tag{45}$$

and corresponds to a modified new quartic term $\tilde{\mathcal{L}}_4 = -\mathcal{L}_2^2$. In spite of the fact that both Lagrangians (42) and (45) lead to the same static solutions, the time-dependent configurations behave differently. In fact, the energy of the second model is not bounded from below, as was already observed in [19]. We comment briefly on this point in subsection 4.5.

### *4.2. $\tilde{M} = 0$, i.e. $\phi^4$ domain walls*

Let us begin our discussion with two relatively simple cases corresponding to two rather special values of the parameters, namely $\tilde{M} = 0$ or $m = 0$.

The first possibility means that the new quartic term is absent and we get the standard case with the $\phi^4$ potential. Of course, the corresponding domain-wall solution is just the $\phi^4$ kink

$$\xi(x) = a \tanh\left[\frac{\sqrt{6}a\lambda}{m}(x + x_0)\right]. \tag{46}$$

### *4.3. $m = 0$, i.e. the quartic $\phi^4$ compactons*

The second special case is more interesting. It is obtained when the parameter $m$ vanishes, $m = 0$, that is, we neglect the standard, quadratic kinetic part of the model. Then we arrive at the following Lagrangian

$$L = \tilde{M}^2|\xi_\mu\xi^\mu|\xi_\mu\xi^\mu - 3\lambda^2(\xi^2 - a^2)^2, \tag{47}$$

In accordance with the previous section we call this model the *quartic $\phi^4$ model*. The pertinent Bogomolny equation reads

$$\xi_x^4 = \frac{\lambda^2}{\tilde{M}^2}(\xi^2 - a^2)^2. \tag{48}$$

It is worth underlining that the same Bogomolny equation can be derived for the model (45) provided $m = 0$, i.e.,

$$L = -\tilde{M}^2(\xi_\mu\xi^\mu)^2 - 3\lambda^2(\xi^2 - a^2)^2, \tag{49}$$

It is obvious that the static sectors of both models are identical. One can solve this equation and get the single compacton solution

$$\xi(x) = \begin{cases} -a & x \leqslant -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ a \sin\sqrt{\frac{\lambda}{\tilde{M}}}x & -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \leqslant x \leqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ a & x \geqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}}, \end{cases} \tag{50}$$

which interpolates between the two distinct vacuum values $-a$ and $a$[3].

[3] Such a compacton solution has been originally found in a non-relativistic model describing a mechanical system of coupled pendulums [36]. It follows from the fact that the static regimes of the model representing a continuous idealization of the system and the quartic sine-Gordon model are identical. On the other hand, sinusoidal solutions to model (49) have been derived in [19]. However, no compacton-like interpretation has been given.

The total energy of the single soliton state is

$$E = 2\lambda^2 a^2 \sqrt{\frac{\tilde{M}}{\lambda}}. \tag{51}$$

Of course, as one expects, the scalar field reaches its vacuum value at a finite distance. There is no exponential tail as in the case of the $\phi^4$ kink. Thus, it is straightforward to generalize it to a solution describing a chain of solitons and anti-solitons with the total topological charge 0 or $\pm 1$. Moreover, in this configuration each constituent soliton (anti-soliton) does not interact with its neighbors. They just do not 'see' each other. Therefore, in such a chain solution the positions of elementary solitons are arbitrary, provided that after each soliton there appears an anti-soliton.

### *4.4. Linear stability*

In this subsection, we shall demonstrate the linear stability of the compactons (50). Here we closely follow the stability analysis of [20]. We introduce general fluctuations around a static (compacton) solution, $\xi(x,t) = \xi(x) + \eta(x,t)$ (here $\xi(x)$ is the compacton solution and $\eta(x,t)$ is the fluctuation field) and insert this expression into the action of a general Lagrangian $L(v,\xi)$ (remember $v \equiv (1/2)\xi^\mu \xi_\mu$). The part of the action quadratic in the fluctuation $\eta$, which is relevant for the stability analysis, is

$$S^{(2)} = \int \mathrm{d}^2x \left( \frac{1}{2} L_v \eta^\mu \eta_\mu + L_{vv} \frac{1}{2} (\xi^\mu \eta_\mu)^2 + L_{\xi\xi} \frac{1}{2} \eta^2 + L_{\xi v} \eta \xi^\mu \eta_\mu \right) \tag{52}$$

or, after using the identity

$$2L_{\xi v} \eta \xi^\mu \eta_\mu = \partial_\mu (L_{\xi v} \eta^2 \xi^\mu) - \eta^2 \partial_\mu (L_{\xi v} \xi^\mu), \tag{53}$$

$$S^{(2)} = \frac{1}{2} \int \mathrm{d}^2x (L_v \eta^\mu \eta_\mu + L_{vv} (\xi^\mu \eta_\mu)^2 + L_{\xi\xi} \eta^2 - \partial_\mu (L_{\xi v} \xi^\mu) \eta^2). \tag{54}$$

The linear equation for the fluctuation field following from this action is

$$\partial_\mu (L_v \eta^\mu + L_{vv} \xi^\mu \xi_\alpha \eta^\alpha) - [L_{\xi\xi} - \partial_\mu (L_{\xi v} \xi^\mu)]\eta = 0. \tag{55}$$

Now we take into account that $\xi$ is static and we replace $v$ by its static limit $w \equiv -(1/2)\xi_x^2$. Further, we assume that

$$\eta(x,t) = \cos(\omega t) \eta(x). \tag{56}$$

The resulting linear ODE for $\eta(x)$ is

$$-\partial_x [(L_w + 2L_{ww} w)\eta_x] - [L_{\xi\xi} + \partial_x (L_{\xi w} \xi_x)]\eta = \omega^2 L_w \eta. \tag{57}$$

For the specific class of Lagrangians $L = F(v) - U(\xi)$ this simplifies to

$$-\partial_x [(F_w + 2F_{ww} w)\eta_x] + U_{\xi\xi} \eta = \omega^2 F_w \eta. \tag{58}$$

Next, we specialize to the Lagrangian (47) such that

$$F = 4\tilde{M}|w|w, \qquad U = 3\lambda^2 (\xi^2 - a^2)^2 \tag{59}$$

and arrive at the equation

$$-12\tilde{M}^2 \partial_x (\xi_x^2 \eta_x) + 12\lambda^2 (3\xi^2 - a^2)\eta = 4\tilde{M}^2 \omega^2 \xi_x^2 \eta. \tag{60}$$

This expression must now be evaluated for the compacton solution (50) for $\xi(x)$. In the outer region of the compacton, i.e., in the region $|x| > \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}}$ where $\xi = \pm a = \text{constant}$, obviously only the solution $\eta = 0$ is possible. As we want $\eta$ to be continuous at the boundary of the compacton, a general $\eta(x)$ may be expressed like

$$\eta(x) = \begin{cases} 0 & x \leqslant -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ \sum_{n=1}^{\infty} b_n \cos n\sqrt{\frac{\lambda}{\tilde{M}}}x & -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \leqslant x \leqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ 0 & x \geqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}}. \end{cases} \tag{61}$$

The restriction to this class of functions will be important in the stability analysis below. Observe that the first derivative of $\eta$ is not continuous at the boundary. This is consistent with the fact that the compacton itself is continuous together with its first derivative. Also, equation (60) is well defined everywhere, because $\eta_x$ is always multiplied by zero at the points of discontinuity. For linear stability, the eigenvalue $\omega^2$ at the rhs of equation (60) has to be positive semi-definite, $\omega^2 \geqslant 0$.

For this to hold, the linear differential operator acting on $\eta$ at the lhs of equation (60) should be a positive semi-definite operator on the space of functions (61). In order to demonstrate this, we rewrite equation (60) like

$$\tilde{H}\eta = 4\tilde{M}^2\omega^2\xi_x^2\eta, \tag{62}$$

where

$$\tilde{H} = -12a^2\tilde{M}^2\lambda\cos^2\sqrt{\frac{\lambda}{\tilde{M}}}x\partial_x^2 + 24a^2\lambda^{\frac{3}{2}}\tilde{M}^{\frac{1}{2}}\sin\sqrt{\frac{\lambda}{\tilde{M}}}x\cos\sqrt{\frac{\lambda}{\tilde{M}}}x\partial_x + 12\lambda^2a^2\left(3\sin^2\sqrt{\frac{\lambda}{\tilde{M}}}x - 1\right). \tag{63}$$

Observe that the operator $\tilde{H}$ maps the space of functions (61) into itself, so its action is well defined on this space. It is useful to introduce the new coordinate $y = \sqrt{\frac{\lambda}{\tilde{M}}}x$ and to rewrite

$$\tilde{H} = 12a^2\lambda^2 H \tag{64}$$

with

$$H = -\cos^2 y\partial_y^2 + 2\sin y\cos y\partial_y + 3\sin^2 y - 1. \tag{65}$$

We now want to demonstrate that $H$ is a positive semi-definite differential operator on the space of functions

$$\eta(y) = \begin{cases} 0 & y \leqslant -\frac{\pi}{2} \\ \sum_{n=1}^{\infty} b_n \cos ny & -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2} \\ 0 & y \geqslant \frac{\pi}{2}. \end{cases} \tag{66}$$

For this it is enough to prove that $H$ is positive semi-definite for all basis functions which are equal to $\cos ny$ inside the compacton region and zero outside. We easily find for $\eta(y) = \cos ny$, after some straightforward calculation,

$$\eta H\eta = \frac{n^2+1}{4}(1+\cos 2ny) + \frac{n^2-3}{4}\cos 2y + \frac{n^2-2n-3}{8}\cos 2(n-1)y + \frac{n^2+2n-3}{8}\cos 2(n+1)y \tag{67}$$

and, therefore,

$$n \neq 1: \qquad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}y\eta H\eta = \frac{n^2+1}{4} > 0 \tag{68}$$

and

$$n = 1: \qquad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}y \eta H \eta = 0, \tag{69}$$

which we wanted to prove. Our compactons are, therefore, indeed stable under small fluctuations. The basis function $\cos y$ for $n = 1$ is a zero mode, which is related to the translational invariance of the compactons (see [20] for a more detailed discussion).

### 4.5. Time dependence

Even though the static Bogomolny solitons are identical in the quartic $\phi^4$ model (both types (47) and (49)) and in the model discussed by Arodź *et al* their dynamics should be different, as the Lagrangians have quite different structure. In the standard case, the dynamics is controlled by the wave equation with a nonlinear (potential) term. It is unlike the quartic $\phi^4$ model where a more complicated structure emerges.

It is not our aim to comprehensively discuss time-dependent configurations, but some interesting remarks can be easily made.

In particular, let us consider the quartic $\phi^4$ model of the type (49) previously discussed in [19, 20]. The full equation of motion is

$$4\tilde{M}^2 \partial_\mu \left[\xi_\nu^2 \xi^\mu\right] + U_\xi = 0. \tag{70}$$

Due to the symmetry between the time and space coordinates, one may find the following solution:

$$\xi(t) = \begin{cases} -a & t \leqslant -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ a \sin\sqrt{\frac{\lambda}{\tilde{M}}} t & -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \leqslant t \leqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ a & t \geqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}}, \end{cases} \tag{71}$$

which represents a smooth transition from one vacuum state to the other, collectively for all $x$. The generalization to a solution representing a composition of such collective jumps is straightforward.

Further, this solution is a solution to the first-order time-dependent differential equation

$$\xi_t^4 = \frac{\lambda^2}{\tilde{M}^2}(\xi^2 - a^2)^2, \tag{72}$$

which can be viewed as a purely time-dependent version of the static Bogomolny equation. Similarly as in the usual case, the solution of this Bogomolny equation gives a topology changing configuration. Here, however, the two distinct vacua are connected by a configuration depending on the time coordinate instead of the space coordinate.

As a consequence of (72), the corresponding energy reads

$$E = \int_{-\infty}^{\infty} L_v \xi_t^2 - L \, \mathrm{d}x = \int_{-\infty}^{\infty} -3\tilde{M}^2 \xi_t^4 + 3\lambda^2 (\xi^2 - a^2)^2 \mathrm{d}x = 0. \tag{73}$$

Thus, this solution is in fact a zero-energy solution[4]. In other words, in this model the global change of the vacuum costs nothing. It is really remarkable that in the model with two separated vacua such a zero-energy transition is possible. This solution might suggest that we have a kind of dynamical restoration of the global $Z_2$ symmetry.

[4] One can observe that while the usual space Bogomolny equation leads to vanishing $T^{11}$, its time-like version enforce $T^{00} = 0$ identically.

Moreover, for a general time-dependent configuration the energy is given by

$$E = \int_{-\infty}^{\infty} -3\tilde{M}^2\xi_t^4 + 2\tilde{M}^2\xi_x^2\xi_t^2 + \tilde{M}^2\xi_x^4 + V\,\mathrm{d}x. \tag{74}$$

Therefore, this expression is not bounded from below and may take arbitrarily large negative values. This might cause stability problems in some applications of the model. In terms of general relativity language, the model does not obey the null-energy condition. It does, however, obey the hyperbolicity condition

$$1 + 2v\frac{L_{vv}}{L_v} > 0 \tag{75}$$

for arbitrary (background) field configurations, and therefore the evolution of small fluctuations will be hyperbolic for general backgrounds. In particular, the linear stability analysis of section 4.4 remains unchanged for this model and, therefore, small fluctuations around the compacton solutions behave the same way in both models. A more detailed discussion of the roll of the null-energy condition and the global hyperbolicity condition in *k*-essence models can be found, e.g., in [37].

The quartic $\phi^4$ model of the type (47) does not possess the zero-energy, topology changing solutions nor has the problem of unbounded energy (of course, it also obeys the hyperbolicity condition (75)). In contrast, the energy is positive definite and reads

$$E = \int_{-\infty}^{\infty} 3\tilde{M}^2|\xi_t^2 - \xi_x^2|\xi_t^2 + \tilde{M}^2|\xi_t^2 - \xi_x^2|\xi_x^2 + V \geqslant 0. \tag{76}$$

Other possible modifications of equation (70) to cure the stability problems mentioned above have been discussed in [19].

### 4.6. Quartic sine-Gordon compactons

Another example worth discussing is the quartic model with the famous sine-Gordon potential [36]

$$L = \tilde{M}^2|\xi_\mu\xi^\mu|\xi_\mu\xi^\mu - \tfrac{3}{2}\lambda^2(1-\cos\xi). \tag{77}$$

Thus, the Bogomolny equation is

$$\xi_x^4 = \frac{\lambda^2}{\tilde{M}^2}(1-\cos\xi). \tag{78}$$

It can be simplified to

$$\xi_x = \sqrt{\frac{\lambda}{\tilde{M}}}\sqrt{\sin\frac{\xi}{2}}. \tag{79}$$

After integration, one can find the simplest compacton solution with unit topological charge in the following exact form (see the appendix for a derivation):

$$\xi(x) = \begin{cases} 0 & x \leqslant x_1\sqrt{\frac{\tilde{M}}{\lambda}} \\ 4\arctan\left[\left(\frac{\mathrm{sn}bx-(1+\sqrt{2})\mathrm{cn}bx}{\mathrm{sn}bx+(1+\sqrt{2})\mathrm{cn}bx}\right)^2\right] & x_1\sqrt{\frac{\tilde{M}}{\lambda}} \leqslant x \leqslant x_2\sqrt{\frac{\tilde{M}}{\lambda}} \\ 2\pi & x \geqslant x_2\sqrt{\frac{\tilde{M}}{\lambda}}. \end{cases} \tag{80}$$

Here

$$b = \frac{1}{4\sqrt{2}(2-\sqrt{2})}\sqrt{\frac{\lambda}{\tilde{M}}}$$

and $x_1, x_2$ are roots (the closest neighbors) of the algebraic equations

$$\mathrm{sn}bx_1 - (1+\sqrt{2})\mathrm{cn}bx_1 = 0, \qquad \mathrm{sn}bx_2 + (1+\sqrt{2})\mathrm{cn}bx_2 = 0.$$

Moreover, sn and cn are the Jacobi elliptic functions. Again, generalization to a multi-compacton configuration is obvious.

### *4.7. Domain-wall solutions in the full model*

Finally, we consider the static soliton solutions of the full model (42). The corresponding Bogomolny equation is

$$3\tilde{M}^2\xi_x^4 + \frac{m^2}{2}\xi_x^2 = 3\lambda^2(\xi^2 - a^2)^2. \tag{81}$$

Although the equation is too complicated to derive an explicit solution, we are able to describe its general properties.

We begin the analysis with the calculation of the asymptotic behavior of the scalar field near the vacuum values. Assuming a smooth approach to the local minima of the potential, that is, $\xi_x' \to 0$ for $x \to \pm\infty$, we get that at sufficiently large distance from the origin $\xi_x'^4 \ll \xi_x'^2$. Thus, the asymptotics is governed by the quadratic term and the exponential tail should be visible for all nonzero values of $m$. Specifically, for $\xi \to \pm a$, one finds

$$\frac{m^2}{2}\xi_x^2 = 3\lambda^2(\xi^2 - a^2)^2 \tag{82}$$

and

$$\xi(x) \simeq \pm a(1 - 2\,\mathrm{e}^{-\frac{2\sqrt{6}a\lambda}{m}x}) \tag{83}$$

for $x \to \pm\infty$. This shows that domain walls for the full model are of the same type as the $\phi^4$ kink. In other words, the exponential tail is a generic feature of topological defects for this model. Therefore, the compacton solution is a really critical case which happens for very special values of the model parameters. In that sense, it is an isolated solution.

While the asymptotics is entirely fixed by the quadratic term, the behavior of the field in the vicinity of the center of the soliton is influenced by the new quartic term. Indeed, one can find that at the origin the solution is given by the following expression:

$$\xi(x) \simeq Ax, \qquad \text{if} \quad x \simeq 0, \tag{84}$$

where

$$A = \pm\frac{m}{\tilde{M}\sqrt{12}}\sqrt{\sqrt{1+\left(\frac{12a\lambda\tilde{M}}{m^2}\right)^2} - 1}.$$

Of course, in general one can find the domain-wall solutions only numerically. Then, from (81) we get the physically relevant root

$$\xi_x^2 = \frac{m^2}{12\tilde{M}^2}\left(\sqrt{1+\left(\frac{12\tilde{M}\lambda}{m^2}\right)^2(\xi^2-a^2)^2} - 1\right). \tag{85}$$

Numerically, finite solutions of this equation have been presented in [20]. In fact, they are smooth functions monotonically interpolating between two distinct vacua.

## 5. Half-compactons

So far we have investigated $k$-deformed models with compactons where the potential term at a local minimum obeys two requirements. Namely

$$U'(\xi_0) = 0, \qquad U''(\xi_0) \neq 0. \tag{86}$$

However, if we allow for more general but still smooth potentials then a new qualitatively different type of solitons can be derived. Specifically, in the case of the quartic models we have to assume that at least at one of the minima the first nonzero derivative is the fourth one. Thus,

$$U'(\xi_0) = U''(\xi_0) = U^{(3)}(\xi_0) = 0, \qquad U^{(4)}(\xi_0) \neq 0. \tag{87}$$

Of course, solutions of the type discussed here can exist in other $k$-modified systems provided a suitable potential is introduced.

### 5.1. Models with half-compactons

The simplest case allowing for half-compactons is provided by a potential with two minima obeying conditions (87) and (86), respectively. The following quartic Lagrangian can serve as a particular example:

$$L = |\xi_\mu \xi^\mu| \xi_\mu \xi^\mu - 3\lambda^4 (1-\xi)^2 (1+\xi)^4. \tag{88}$$

Then the static equation reads

$$\xi_x^4 = \lambda^4 (1-\xi)^2 (1+\xi)^4 \tag{89}$$

and we find the solution

$$\xi = 1 - 2\tanh^2 \frac{\lambda(x-x_0)}{\sqrt{2}}. \tag{90}$$

This standard static configuration (note the exponential tails at $x \to \pm\infty$) is well defined for all $x$ and describes a topologically trivial solution joining the vacuum $\xi = -1$ with itself. However, one can observe that it is a collection of a *half-compacton* and an *anti half-compacton* glued at $x = x_0$, where the second vacuum $\xi = 1$ is reached.

The half-compacton goes from the vacuum $\xi = 1$ to $\xi = 0$ and is given by the formula

$$\xi = \begin{cases} 1 & x \leqslant x_0 \\ 1 - 2\tanh^2 \dfrac{\lambda(x-x_0)}{\sqrt{2}} & x > x_0. \end{cases} \tag{91}$$

The construction of the anti half-compacton is straightforward (it takes the vacuum value $\xi = 1$ for $x \geqslant x_0$). The name half-compacton is also obvious. It denotes an object which reaches its two different vacuum values in two different ways. At one end of the defect (where the potential has a nonzero second derivative) the approach is power-like (here parabolic). In other words, at this end the soliton behaves like a compacton. On the other hand, at the second end the exponential tail emerges and the solution looks like a standard kink. In our example, the parabolic approach occurs in the neighborhood of the vacuum at $\xi = 1$,

$$\xi \simeq 1 - \lambda^2 x^2, \qquad \text{for} \quad x \simeq 0, \tag{92}$$

whereas the vacuum at $\xi = -1$ is approached exponentially,

$$\xi \simeq -1 + A\,\mathrm{e}^{-\lambda\sqrt{2}x}, \qquad \text{for} \quad x \to \infty. \tag{93}$$

The energy of the half-compacton configuration is

$$E = \frac{4^5}{2\sqrt{2}} \frac{16}{1155} \lambda^3. \tag{94}$$

A slightly more complicated situation occurs if the quartic Lagrangian possesses a potential with three vacua

$$L = |\xi_\mu \xi^\mu| \xi_\mu \xi^\mu - 3\lambda^4 \xi^4 (1 - \xi^2)^2. \tag{95}$$

Here, the minima at $\xi = \pm 1$ are of type (87) whereas the minimum at $\xi = 0$ satisfies condition (86). One can easily obtain the solution

$$\pm \ln \left| \frac{1 + \sqrt{1 - \xi^2}}{\xi} \right| = \lambda(x - x_0) \tag{96}$$

or

$$\xi = \pm \frac{2\, \mathrm{e}^{\pm\lambda(x - x_0)}}{1 + \mathrm{e}^{\pm 2\lambda(x - x_0)}}. \tag{97}$$

This soliton configuration (with exponential tails) is well defined for all $x$ and describes a solution joining the three vacua $-1, 0, +1$. Again, it is a collection of two half-compactons glued at $x = x_0$.

The first half-compacton joins the vacua $\xi = 1$ and $\xi = 0$

$$\xi = \begin{cases} 1 & x \leqslant x_0 \\ \dfrac{2\, \mathrm{e}^{-\lambda(x - x_0)}}{1 + \mathrm{e}^{-2\lambda(x - x_0)}} & x > x_0. \end{cases} \tag{98}$$

The second type of half-compacton connects $\xi = -1$ with $\xi = 0$

$$\xi = \begin{cases} -1 & x \leqslant x_0 \\ \dfrac{-2\, \mathrm{e}^{-\lambda(x - x_0)}}{1 + \mathrm{e}^{-2\lambda(x - x_0)}} & x > x_0. \end{cases} \tag{99}$$

Both half-compactons have the energy

$$E = \frac{2}{35} \lambda^3. \tag{100}$$

In the last example, we consider a model with infinitely many vacua of both types (87) and (86),

$$L = |\xi_\mu \xi^\mu| \xi_\mu \xi^\mu - 3(1 - \cos \xi)(1 + \cos \xi)^2. \tag{101}$$

The vacua with exponential approach are located at $\xi = (2k + 1)\pi$, whereas the vacua at $\xi = 2k\pi$ are of the compacton type and are reached at a finite distance. The static half-compacton solution corresponding to the transition between vacua $2k\pi$ and $(2k + 1)\pi$ reads

$$\xi = \begin{cases} 0 & x \leqslant x_0 \\ \tilde{\xi} & x > x_0, \end{cases} \tag{102}$$

where $\tilde{\xi}$ is a solution to the equation

$$\arctan \sqrt{\left| \sin \frac{\xi}{2} \right|} + \arctan \sqrt{\left| \sin \frac{\xi}{2} \right|} = \pm \frac{\sqrt[4]{8}(x - x_0)}{2}. \tag{103}$$

Of course, in equation (102) the scalar field is determined modulo $2k\pi$ only.

*5.2. Models with compactons and half-compactons*

Finally, let us study a model with a four vacua potential

$$L = |\xi_\mu \xi^\mu| \xi_\mu \xi^\mu - 3(1 - \xi^2)^2 (4 - \xi^2)^4. \tag{104}$$

Here, the minima at $\xi = \pm 1$ are of the type (86) and the minima at $\xi = \pm 2$ are of the type (87). Therefore, one might expect that this system allows for a compacton (joining $-1$ with $+1$) and two types of half-compactons. After some calculation one finds the simplest compacton configuration

$$\xi = \begin{cases} -1 & x - x_0 \leqslant -x_1 \\ \tilde{\xi} & -x_1 < x - x_0 < x_1 \\ 1 & x - x_0 \geqslant x_1, \end{cases} \tag{105}$$

where $\tilde{\xi}$ is a solution of the following expression for $-1 \leqslant \tilde{\xi} \leqslant 1$:

$$\frac{\xi(\sqrt{1-\xi^2} - 1)}{\xi^2 + \sqrt{1-\xi^2} - 1} = \pm \frac{2}{\sqrt{3}} \tan 2\sqrt{3}(x - x_0). \tag{106}$$

and the joining point $x_1$ is

$$x_1 = \frac{\pi}{4\sqrt{3}}.$$

Due to the complexity of equation (106) we are not able to derive the explicit expression for the scalar field. Nonetheless, the asymptotic behavior can be easily obtained

$$\xi \simeq \pm 1 \mp \frac{3^2}{2} \left( x \mp x_0 \mp \frac{\pi}{4\sqrt{3}} \right)^2, \qquad \text{for} \qquad x \mp x_0 \mp \frac{\pi}{4\sqrt{3}} \simeq 0. \tag{107}$$

The half-compactons are given by

$$\xi^+ = \begin{cases} 1 & x - x_0 \leqslant x_2 \\ \tilde{\xi}^+ & x - x_0 > x_2, \end{cases} \tag{108}$$

$$\xi^- = \begin{cases} -1 & x - x_0 \leqslant x_2 \\ \tilde{\xi}^- & x - x_0 > x_2, \end{cases} \tag{109}$$

where $\tilde{\xi}^\pm$ are the positive and negative solutions of the equation (here $1 \leqslant \tilde{\xi}^+ \leqslant 2$ and $-2 \leqslant \tilde{\xi}^- \leqslant -1$)

$$(7 - 4\sqrt{3}) \ln|(\sqrt{\xi^2 - 1} + \xi)^2 - (7 - 4\sqrt{3})| \\ + (7 + 4\sqrt{3}) \ln|(\sqrt{\xi^2 - 1} + \xi)^2 - (7 + 4\sqrt{3})| = \pm 4\sqrt{3}(x - x_0)$$

and

$$x_2 = \frac{1}{4\sqrt{3}} \ln \frac{(4\sqrt{3} + 6)^{7+4\sqrt{3}}}{(4\sqrt{3} - 6)^{7-4\sqrt{3}}}.$$

Here, once again one finds the parabolic approach to the vacua $\pm 1$. The exponential tail emerges if the scalar field tends to the vacua $\pm 2$.

Let us make some general observations concerning topological defects in the quartic Lagrangians. If two nearest minima of the potential are of the same kind we get either a compacton or a standard soliton. For potentials with $U''(\xi_0) \neq 0$ we get compactons whereas for $U'(\xi_0) = U''(\xi_0) = U^{(3)}(\xi_0) = 0$ and $U^{(4)}(\xi_0) \neq 0$ we find kinks with an exponential tail. If the closest vacua are of distinct types, then a half-compacton occurs.

It is not difficult to convince oneself that half-compactons may be derived also in models with the usual, quadratic kinetic part. Then the corresponding potential should have mixed $UV$ shape.

## 6. Conclusions

We think that the main achievement of the paper is the demonstration of the close relation between the existence of compactons and the appearance of a nonstandard kinetic term in the Lagrangian. In fact, the compactons emerge as a result of the specific relation between the kinetic and potential parts of the Lagrangian. Thus, this kind of topological defects is no longer one-to-one connected with $V$-shaped potentials. In contrast, they can be found for well behaved, analytical potentials if a nonstandard kinetic term is allowed. Of course, this strongly broadens the range of possible situations where such objects can be relevant, indicating that the notion of compactons might have even more interesting applications than expected previously.

Moreover, we demonstrated that compactons do not necessarily reach their vacuum value in the parabolic way. Depending on the particular form of the kinetic term one can obtain arbitrary power-like approaches. Similarly, the specific form of the approximate scaling symmetry is also fixed by the mutual relationship between the kinetic and potential parts of the action.

As an example, we established the existence of compactons as critical domain-wall solutions in a generalized Skyrme model. Here, the important point was that a model consisting of a real scalar field $\xi$ with a quartic kinetic term and a regular potential (e.g., the quartic $\phi^4$ model) could be embedded into the generalized Skyrme model, therefore compacton solutions will exist more generally for theories where such an embedding is possible.

We also established the existence of half-compactons, that is, of soliton solutions which approach some vacua at a finite distance (like a compacton), whereas other vacua are approached in the kink-like fashion (with an exponential tail).

Let us also mention that models with a non-canonical kinetic term, which in principle can possess compacton solutions, have been widely discussed in many contexts. A class of models for which one can easily construct compacton solutions is the following:

$$L = V(\xi)F(v). \tag{111}$$

Such Lagrangians are considered as alternative models for a canonical quintessence field. Moreover, this form of the Lagrangian is suggested by string theory [38–42]. The pertinent static Bogomolny equation reads

$$V(\xi)(F - 2F_w w) = 0 \tag{112}$$

and possesses solutions in the form $w = -a_i^2/2$, $a_i^2 \geqslant 0$ [20]. Thus

$$\xi = a_i x. \tag{113}$$

This solution represents a compacton if the energy density is localized in a finite region. Thus, we may assume that

$$V(\xi) = \begin{cases} 0 & \xi < -\xi_0 \\ V_0 & -\xi_0 \leqslant \xi \leqslant \xi_0 \\ 0 & \xi > \xi_0. \end{cases} \tag{114}$$

Then, the corresponding energy

$$E = -F(w = -a_i^2/2) \int_{-\infty}^{\infty} V(\xi)\, \mathrm{d}x = -\frac{F(w = -a_i^2/2)}{|a_i|} \int_{-\xi_0}^{\xi_0} V(\xi)\, \mathrm{d}\xi \tag{115}$$

is finite. For a detailed discussion of such a family of models, see [20].

There are several directions in which the present work may be continued. For example, dynamical properties of the half-compacton can be analyzed. As they are asymmetric topological defects, their interaction should depend on the mutual orientation of the ends. It can be done at least in the simplest (and best understood) case of the standard kinetic term and a $UV$-type potential. Another issue worth investigating is a generalization of the inverse scattering method (or other methods known from the standard soliton theory) for $k$-deformed Lagrangians.

We believe that the established relations between compactons and various $k$-defects (or more general fields with a nonstandard static gradient term) will open a new window for the application of compactons.

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## Appendix

Here we calculate the quartic SG compacton. From (79) we get

$$2\int \frac{\mathrm{d}y}{\sqrt{\sin y}} = x - x_0, \tag{A.1}$$

where $y = \xi/2$. After introducing $z^2 = \tan(y/2)$ we find

$$\int \frac{\mathrm{d}z}{\sqrt{1+z^4}} = \frac{1}{4\sqrt{2}}(x - x_0). \tag{A.2}$$

This integral can be related to the elliptic integral of the first type. Indeed, if we substitute

$$z = \frac{\tan\phi - (1+\sqrt{2})}{\tan\phi + (1+\sqrt{2})}, \tag{A.3}$$

then we obtain

$$(2-\sqrt{2})\int \frac{\mathrm{d}\phi}{1 - k^2\sin^2\phi} = \frac{1}{4\sqrt{2}}(x - x_0), \tag{A.4}$$

where

$$k^2 = \frac{4\sqrt{2}}{3+2\sqrt{2}}.$$

This expression can be inverted

$$\phi = \mathrm{am}\left[\frac{1}{4\sqrt{2}(2-\sqrt{2})}(x - x_0)\right]. \tag{A.5}$$

Now, inserting it into (A.3) and using sin am ≡ sn, cos am ≡ cn we arrive at the result.

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# Corrigendum

**k-defects as compactons**

Adam C, Sanchez-Guillen J and Wereszczynski A 2007 *J. Phys. A: Math. Theor.* **40** 13625 (arXiv:0705.3554)

In Adam *et al* (2007 *J. Phys. A: Math. Theor.* **40** 13625), a compact soliton solution has been constructed and its stability under linear fluctuations has been proved. The stability proof in section 4.4 of this paper is, however, incorrect. In this corrigendum we provide the correct proof.

In [1] an explicit solution of a compact topological soliton has been given, and in section 4.4 of this paper the stability of the compact soliton under linear fluctuations has been proved. The proof of stability in this paper is, however, erroneous. The statement itself is correct (i.e., the compact soliton is stable under linear fluctuations), and the correct proof is provided in the following. As the error affects most of section 4.4, we prefer to simply rewrite this section, so the text below just is the new, correct version of section 4.4 of [1].

In this corrected version of subsection 4.4 of [1] we shall demonstrate the linear stability of the compactons of section 4.3 of this reference. Here we closely follow the stability analysis of [2]. We introduce general fluctuations around a static (compacton) solution, $\xi(x,t) = \xi(x) + \eta(x,t)$ (here, $\xi(x)$ is the compacton solution and $\eta(x,t)$ is the fluctuation field), and insert this expression into the action of a general Lagrangian $L(v,\xi)$ (remember $v \equiv (1/2)\xi^\mu\xi_\mu$). The part of the action quadratic in the fluctuation $\eta$, which is relevant for the stability analysis, is

$$S^{(2)} = \int \mathrm{d}^2x \left( \frac{1}{2} L_v \eta^\mu \eta_\mu + L_{vv} \frac{1}{2} (\xi^\mu \eta_\mu)^2 + L_{\xi\xi} \frac{1}{2} \eta^2 + L_{\xi v} \eta \xi^\mu \eta_\mu \right) \tag{1}$$

or, after using the identity

$$2L_{\xi v} \eta \xi^\mu \eta_\mu = \partial_\mu (L_{\xi v} \eta^2 \xi^\mu) - \eta^2 \partial_\mu (L_{\xi v} \xi^\mu), \tag{2}$$

$$S^{(2)} = \frac{1}{2} \int \mathrm{d}^2x (L_v \eta^\mu \eta_\mu + L_{vv} (\xi^\mu \eta_\mu)^2 + L_{\xi\xi} \eta^2 - \partial_\mu (L_{\xi v} \xi^\mu) \eta^2). \tag{3}$$

The linear equation for the fluctuation field following from this action is

$$\partial_\mu (L_v \eta^\mu + L_{vv} \xi^\mu \xi_\alpha \eta^\alpha) - [L_{\xi\xi} - \partial_\mu (L_{\xi v} \xi^\mu)]\eta = 0. \tag{4}$$

Now we take into account that $\xi$ is static, and we replace $v$ by its static limit $w \equiv -(1/2)\xi_x^2$. Further, we assume that

$$\eta(x,t) = \cos(\omega t)\eta(x). \tag{5}$$

The resulting linear ODE for $\eta(x)$ is

$$-\partial_x [(L_w + 2L_{ww} w)\eta_x] - [L_{\xi\xi} + \partial_x (L_{\xi w} \xi_x)]\eta = \omega^2 L_w \eta. \tag{6}$$

For the specific class of Lagrangians $L = F(v) - U(\xi)$ this simplifies to

$$-\partial_x[(F_w + 2F_{ww}w)\eta_x] + U_{\xi\xi}\eta = \omega^2 F_w \eta. \tag{7}$$

Next, we specialize to the Lagrangian of section 4.3 of [1]

$$F = 4\tilde{M}^2|w|w, \qquad U = 3\lambda^2(\xi^2 - a^2)^2, \tag{8}$$

and arrive at the equation

$$-12\tilde{M}^2\partial_x\left(\xi_x^2\eta_x\right) + 12\lambda^2(3\xi^2 - a^2)\eta = 4\tilde{M}^2\omega^2\xi_x^2\eta. \tag{9}$$

This expression must now be evaluated for the compacton solution $\xi(x)$ of section 4.3 of [1]. In the outer region of the compacton, i.e., in the region $|x| > \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}}$ where $\xi = \pm a =$ const., obviously only the solution $\eta = 0$ is possible. As we want $\eta$ to be continuous at the boundary of the compacton, a general $\eta(x)$ should go to zero at the compacton boundaries. The corresponding space of functions may be divided into an even and an odd subspace under the reflection $x \to -x$, and basis functions for the two subspaces are

$$\eta_n(x) = \begin{cases} 0 & x \leqslant -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ \cos(2n+1)\sqrt{\frac{\lambda}{\tilde{M}}}x & -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \leqslant x \leqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ 0 & x \geqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \end{cases} \tag{10}$$

for the even subspace (here $n = 0, \ldots, \infty$) and

$$\zeta_m(x) = \begin{cases} 0 & x \leqslant -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ \sin 2m\sqrt{\frac{\lambda}{\tilde{M}}}x & -\frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \leqslant x \leqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \\ 0 & x \geqslant \frac{\pi}{2}\sqrt{\frac{\tilde{M}}{\lambda}} \end{cases} \tag{11}$$

for the odd subspace (here $m = 1, \ldots, \infty$). The restriction on this class of functions will be important in the stability analysis below. Observe that the first derivative of $\eta$ is not continuous at the boundary. This is consistent with the fact that the compacton itself is continuous together with its first derivative. Also, equation (9) is well defined everywhere, because $\eta_x$ is always multiplied by zero at the points of discontinuity.

For linear stability, the eigenvalue $\omega^2$ on the rhs of equation (9) has to be positive semi-definite, $\omega^2 \geqslant 0$. For this to hold, the linear differential operator acting on $\eta$ on the lhs of equation (9) should be a positive semi-definite operator on the space of functions (10) and (11). In order to demonstrate this, we rewrite equation (9) as

$$\tilde{H}\eta = 4\tilde{M}^2\omega^2\xi_x^2\eta, \tag{12}$$

where

$$\tilde{H} = -12a^2\tilde{M}\lambda\cos^2\sqrt{\frac{\lambda}{\tilde{M}}}x\,\partial_x^2 + 24a^2\lambda^{\frac{3}{2}}\tilde{M}^{\frac{1}{2}}\sin\sqrt{\frac{\lambda}{\tilde{M}}}x\cos\sqrt{\frac{\lambda}{\tilde{M}}}x\,\partial_x + 12\lambda^2a^2\left(3\sin^2\sqrt{\frac{\lambda}{\tilde{M}}}x - 1\right). \tag{13}$$

It is useful to introduce the new coordinate $y = \sqrt{\frac{\lambda}{\tilde{M}}}x$ and to rewrite

$$\tilde{H} = 12a^2\lambda^2 H \tag{14}$$

with

$$H = -\cos^2 y\partial_y^2 + 2\sin y\cos y\partial_y + 3\sin^2 y - 1. \tag{15}$$

We now want to demonstrate that the operator $H$ is positive semi-definite on the space of functions which are zero for $|y| \geqslant \frac{\pi}{2}$ and continuous at the compacton boundaries $y = \pm\frac{\pi}{2}$. This space may be divided into an even and an odd subspace under the reflection $y \to -y$, and these two subspaces may be treated separately, because the operator $H$ is even and does not mix the two subspaces. A basis for the even subspace is (here $n = 0, \ldots, \infty$)

$$\eta_n(y) = \begin{cases} 0 & y \leqslant -\frac{\pi}{2} \\ \cos(2n+1)y & -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2} \\ 0 & y \geqslant \frac{\pi}{2}, \end{cases} \tag{16}$$

whereas a basis for the odd subspace is (here $m = 1, \ldots, \infty$)

$$\zeta_m(y) = \begin{cases} 0 & y \leqslant -\frac{\pi}{2} \\ \sin 2my & -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2} \\ 0 & y \geqslant \frac{\pi}{2}. \end{cases} \tag{17}$$

We remark that the (correct) basis functions displayed here differ from the (incorrect) ones in [1], see equations (61) and (66) of [1], which is the first error in this reference.

Next, we want to prove the positive semi-definiteness of $H$ on the two subspaces. For the even subspace we find

$$\begin{aligned} \cos(2m+1)yH\cos(2n+1)y &= \left(n^2+n+\tfrac{1}{2}\right)[\cos 2(m-n)y + \cos(2(m+n+1)y] \\ &\quad + \tfrac{1}{2}(n^2-1)[\cos 2(m-n+1)y + \cos 2(m+n)y] \\ &\quad + \tfrac{1}{2}(n+2n)[\cos 2(m-n-1)y + \cos 2(m+n+2)y] \end{aligned} \tag{18}$$

and, therefore,

$$\begin{aligned} \langle m|H|n\rangle &\equiv \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}y\cos(2m+1)yH\cos(2n+1)y \\ &= \pi\left[\left(n^2+n+\frac{1}{2}\right)(\delta_{m,n}-\delta_{m,0}\delta_{n,0}) + \frac{1}{2}(n^2-1)\delta_{m,n-1} + \frac{1}{2}(n^2+2n)\delta_{m,n+1}\right]. \end{aligned} \tag{19}$$

It obviously holds that $\langle n|H|n\rangle \geqslant 0$ for all $n$, but this is only a necessary and not a sufficient condition for the positive semi-definiteness of $H$. This is the second error of [1].

We have to demonstrate positive semi-definiteness for a general vector

$$|v\rangle = \sum_{n=0}^{\infty} c_n\cos(2n+1)y, \tag{20}$$

where, however, we will restrict to normalizable vectors $v$. A normalizable vector may always be approximated to arbitrary precision by a vector

$$|v\rangle = \sum_{n=0}^{N} c_n \cos(2n+1)y \tag{21}$$

for sufficiently large but finite $N$; therefore, we will restrict to this case in the following.

Before continuing, we remark that the basis function $\cos y$ for $n = 0$ is a zero mode of the operator $H$, which is related to the translational invariance of the compactons, see [2] for a more-detailed discussion. Therefore, all matrix elements with $m = 0$ or $n = 0$ are zero, and we may assume $c_0 = 0$ without loss of generality. Taking this fact into account, we find

$$\begin{aligned}
\langle v|H|v\rangle &= \sum_{m,n=1}^{N} c_n \bar{c}_m \langle m|H|n\rangle \\
&= \pi \sum_{n=1}^{N} \left[ c_n \bar{c}_n \left( n^2 + n + \frac{1}{2} \right) + c_n \bar{c}_{n-1} \frac{1}{2}(n^2 - 1) + c_n \bar{c}_{n+1} \frac{1}{2}(n^2 + 2n) \right] \\
&\geqslant \pi \sum_{n=1}^{N} \left[ c_n \bar{c}_n \left( n^2 + n + \frac{1}{2} \right) - |c_n||\bar{c}_{n-1}| \frac{1}{2}(n^2 - 1) - |c_n||\bar{c}_{n+1}| \frac{1}{2}(n^2 + 2n) \right] \\
&= \pi \sum_{n=1}^{N} \left[ |c_n|^2 \left( n^2 + n + \frac{1}{2} \right) - |c_n||c_{n-1}|(n^2 - 1) \right],
\end{aligned} \tag{22}$$

where $c_{N+1} \equiv 0$ by assumption. We now want to prove that the above expression is positive semi-definite. The positive semi-definiteness of this expression is implied by the inequality

$$\sum_{n=1}^{N} [|c_n|^2 - |c_n||c_{n-1}|] \geqslant 0 \tag{23}$$

because of the inequality

$$n^2 + n + \tfrac{1}{2} \geqslant n^2 - 1. \tag{24}$$

Finally, inequality (23) may be proved easily with the help of Hoelder's inequality.

In fact, Hoelder's inequality reads

$$\left| \sum_{n=1}^{N} a_n b_n \right| \leqslant \left( \sum_{n=1}^{N} |a_n|^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{N} |b_n|^q \right)^{\frac{1}{q}}, \tag{25}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{26}$$

Now we set $p = q = 2$ and $a_n = |c_n|$, $b_n = |c_{n-1}|$ and obtain

$$\sum_{n=1}^{N} |c_n||c_{n-1}| \leqslant \left( \sum_{n=1}^{N} |c_n|^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{N} |c_{n-1}|^2 \right)^{\frac{1}{2}}. \tag{27}$$

Further we have

$$\left( \sum_{n=1}^{N} |c_{n-1}|^2 \right)^{\frac{1}{2}} = \left( \sum_{n=0}^{N-1} |c_n|^2 \right)^{\frac{1}{2}} \leqslant \left( \sum_{n=1}^{N} |c_n|^2 \right)^{\frac{1}{2}}, \tag{28}$$

where we used $c_0 = 0$. Inserting this last inequality into (27) just gives the inequality (23), which we wanted to prove.

The proof for the odd subspace (17) is completely analogous. Indeed, we find

$$\begin{aligned} \sin 2myH \sin 2ny = {} & \left(n^2 + \tfrac{1}{4}\right)[\cos 2(m-n)y - \cos(2(m+n)y] \\ & + \tfrac{1}{2}\left(n^2 + n + \tfrac{3}{4}\right)[\cos 2(m-n-1)y - \cos 2(m+n+1)y] \\ & + \tfrac{1}{2}\left(n^2 - n + \tfrac{3}{4}\right)[\cos 2(m-n+1)y - \cos 2(m+n-1)y] \end{aligned} \tag{29}$$

and, therefore,

$$\begin{aligned} \langle m|H|n\rangle \equiv {} & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}y \sin 2myH \sin 2ny \\ = {} & \pi\left[\left(n^2 + \frac{1}{4}\right)\delta_{m,n} + \frac{1}{2}\left(n^2 + n + \frac{3}{4}\right)\delta_{m,n+1} + \frac{1}{2}\left(n^2 - n + \frac{3}{4}\right)\delta_{m,n-1}\right]. \end{aligned} \tag{30}$$

For a general vector

$$|v\rangle = \sum_{n=0}^{N} c_n \sin 2ny, \tag{31}$$

we therefore obtain

$$\begin{aligned} \langle v|H|v\rangle = {} & \sum_{m,n=1}^{N} c_n \bar{c}_m \langle m|H|n\rangle \\ = {} & \pi \sum_{n=1}^{N}\left[c_n\bar{c}_n\left(n^2 + \frac{1}{4}\right) + c_n\bar{c}_{n-1}\frac{1}{2}\left(n^2 - n + \frac{3}{4}\right) + c_n\bar{c}_{n+1}\frac{1}{2}\left(n^2 + n + \frac{3}{4}\right)\right] \\ \geqslant {} & \pi \sum_{n=1}^{N}\left[c_n\bar{c}_n\left(n^2 + \frac{1}{4}\right) - |c_n||\bar{c}_{n-1}|\frac{1}{2}\left(n^2 - n + \frac{3}{4}\right) - |c_n||\bar{c}_{n+1}|\frac{1}{2}\left(n^2 + n + \frac{3}{4}\right)\right] \\ = {} & \pi \sum_{n=1}^{N}\left[|c_n|^2\left(n^2 + \frac{1}{4}\right) - |c_n||c_{n-1}|\left(n^2 - n + \frac{3}{4}\right)\right]. \end{aligned} \tag{32}$$

Using the inequality

$$n^2 + \tfrac{1}{4} \geqslant n^2 - n + \tfrac{3}{4}, \tag{33}$$

the positive semi-definiteness of expression (32) is again implied by the inequality (23), which has been proved above.

Finally, let us mention that alternative stability proofs for the compact soliton of section 4.3 of [1], somewhat different in spirit than the one presented here, have been given recently in [3] (see section 6 of this reference) and [4].

## References

[1] Adam C, Sanchez-Guillen J and Wereszczynski A 2007 *J. Phys. A: Math. Theor.* **40** 13625 (arXiv:0705.3554)
[2] Bazeia D, Losano L, Menezes R and Oliveira J C R E 2007 *Eur. Phys. J.* C **51** 953 (arXiv:hep-th/0702052)
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[4] Bazeia D, Losano L and Menezes R 2008 *Phys. Lett.* B **668** 246 (arXiv:0807.0213)